

A Review of Fixed Point Theory in E-Fuzzy Metric Spaces

Megha Chourasia¹, Uday Dolas²

¹Research Scholar, Department of Mathematics, Barkatullah University, Bhopal (M.P.), megha24061996@gmail.com

India

²Supervisor, Department of Mathematics, Barkatullah University, Bhopal (M.P.), India

Abstract – Fixed point theory supplies a rigorous framework for establishing existence, uniqueness, approximation and stability of solutions of nonlinear equations. Its classical development begins with Banach's contraction principle and extends through Kannan, Chatterjea, Reich, Zamfirescu, Hardy-Rogers and other generalized contractions. The emergence of fuzzy set theory and fuzzy metric spaces subsequently made it possible to analyze nearness, convergence and iterative behavior when distance information is uncertain or graded rather than crisp. This review critically examines that development and focuses on the relatively recent literature on E-fuzzy metric spaces. Particular attention is given to structural properties, convergence and E-Cauchy sequences, completeness, Banach-type, Kannan-type and Chatterjea-type contractions, common fixed points, compatibility and weak compatibility, stability, well-posedness and nonlinear applications. The review also emphasizes a terminological issue: the expression "E-fuzzy metric space" is not completely uniform in the literature. One important strand associated with Sukanya and Jose develops a generalized, three-variable fuzzy metric structure related to G-metric ideas, whereas Mecheraoui et al. use the same label for a particular class of George-Veeramani fuzzy metric spaces with an additional uniformity property. Accordingly, the axiom system must be fixed explicitly before new theorems are stated. Recent E-fuzzy studies on contraction theorems, coupled fixed points, tripled fixed points and common fixed points indicate that the area remains open for systematic comparison of contractive classes, assumption reduction, stability and well-posedness. The review concludes by mapping these gaps to the objectives and hypotheses of the proposed thesis and by identifying a coherent program for new theorem development and applications.

Keywords: E-fuzzy metric space; fuzzy metric space; fixed point theorem; Banach contraction; Kannan contraction; Chatterjea contraction; common fixed point; compatibility; weak compatibility; E-Cauchy sequence; completeness; stability; well-posedness; nonlinear analysis.

I. INTRODUCTION

Fixed point theory is one of the most productive techniques in nonlinear analysis because many equations can be reformulated as the problem of finding a point that is invariant under an operator. If T is a self-mapping of a nonempty set X , a point x^* is a fixed point when $T(x^*) = x^*$. This apparently simple relation becomes powerful when the ambient space has enough structure to force existence and uniqueness and when an iterative process provides a computable approximation. Banach's contraction principle [1] established the basic paradigm: completeness of the metric space together with a strict contraction guarantees a unique fixed point and convergence of Picard iteration. The theorem subsequently became a template for nonlinear equations, differential and integral equations, approximation theory and optimization.

A major direction in fixed point research has been to weaken the contraction assumption without losing the fixed point conclusion. Kannan [2] replaced the direct distance between x and y by distances from each point to its image. Reich [3] introduced a weighted combination of several distance terms, while Chatterjea [4] used cross-distances between each point and the image of the other. Zamfirescu [5] unified several classical alternatives, Hardy and Rogers [6] broadened the weighted structure, and Rhoades [8] compared numerous contractive conditions. These developments are highly relevant to E-fuzzy metric spaces because the thesis seeks Banach-, Kannan- and Chatterjea-type results under generalized fuzzy nearness.

The second line of development concerns uncertainty. Zadeh's fuzzy set theory [9] replaced binary membership by a graded membership function, thereby providing a systematic mathematical language for vagueness. Kramosil and Michalek [10] developed fuzzy metric ideas, and

George and Veeramani [11,12] supplied a widely used formulation based on continuous triangular norms. Fuzzy metric fixed point theory then emerged through work such as Grabiec [13], Gregori and Sapena [14], Vasuki [15] and Mihet [16]. The supplied thesis draft identifies this progression as the immediate foundation for a systematic study of E-fuzzy metric spaces, with explicit emphasis on convergence, Cauchy sequences, completeness, generalized contractions, common fixed points, stability and applications [9-16].

The present review therefore has three purposes. First, it organizes the classical and fuzzy fixed point literature around the mechanisms needed for E-fuzzy theorem development. Second, it critically compares the assumptions used in generalized fixed point theory and identifies where E-fuzzy results can reasonably be expected to improve existing fuzzy metric results. Third, it converts the supplied research objectives and hypotheses into a concrete theorem-building agenda, including definitions, proof strategy, examples, common fixed points, stability, well-posedness and nonlinear applications.

II. SCOPE AND METHOD OF THE REVIEW

This paper is a focused narrative and conceptual review rather than a systematic bibliometric survey. The literature base combines the classical references and fuzzy metric references assembled in the thesis draft with published work explicitly concerned with generalized E-fuzzy metric spaces. The most relevant E-fuzzy sources include Sukanya and Jose's generalized fuzzy metric and fixed point work [25,27], their contraction theorem in E-fuzzy metric space [28], Mecheraoui et al.'s distinct E-fuzzy framework [26], Shahana and Jose's coupled and tripled fixed point studies

[29,30], and Kumar and Manjeet's generalized E-fuzzy b-metric common fixed point results [31].

The review is organized by mathematical role rather than by publication year alone. Foundational definitions are considered first; generalized contraction mechanisms are then compared; common fixed point interaction conditions are examined; and finally stability, well-posedness, applications and research gaps are discussed. This arrangement is appropriate for the thesis because it directly links the literature to theorem design and hypothesis testing.

III. MATHEMATICAL FOUNDATIONS

3.1 Fixed points, iterative sequences and completeness

Let (X, d) be a metric space and $T: X \rightarrow X$. A point $x^* \in X$ is a fixed point of T if $T(x^*) = x^*$. Starting from $x_0 \in X$, the Picard iteration is defined by

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, \dots \quad (1)$$

The central proof strategy in many contraction theorems is to show that $\{x_n\}$ is Cauchy, invoke completeness to obtain a limit x^* , and then show that the limit satisfies $T(x^*) = x^*$. Uniqueness is normally proved by assuming two fixed points and applying the contraction inequality to them. The same architecture persists in fuzzy and E-fuzzy settings, although the meaning of "Cauchy", the limiting relation and the form of the contraction must be adapted to the chosen fuzzy metric axioms.

3.2 Banach contraction principle

Banach's contraction condition is expressed by the inequality

$$d(Tx, Ty) \leq \lambda d(x, y), \quad 0 \leq \lambda < 1. \quad (2)$$

On a complete metric space this condition implies a unique fixed point and geometric convergence of the Picard sequence [1]. Its importance for the present study is methodological: an E-fuzzy Banach theorem should identify an appropriate fuzzy analogue of strict distance reduction and then prove that this reduction drives E-Cauchy behavior and uniqueness.

3.3 Kannan-type contraction

Kannan's condition does not compare $d(Tx, Ty)$ directly with $d(x, y)$. Instead it uses the displacement of each point under T :

$$d(Tx, Ty) \leq \alpha [d(x, Tx) + d(y, Ty)], \quad 0 \leq \alpha < 1/2. \quad (3)$$

This is significant because Kannan-type theorems can apply to mappings that fail to be Banach contractions [2]. In E-fuzzy research, the analogue should preserve this "self-displacement" structure. A theorem merely replacing d by a fuzzy symbol without reflecting the logic of (3) would not constitute a meaningful Kannan generalization.

3.4 Chatterjea-type contraction

The Chatterjea condition is based on cross-distances:

$$d(Tx, Ty) \leq \beta [d(x, Ty) + d(y, Tx)], \quad 0 \leq \beta < 1/2. \quad (4)$$

The cross-term mechanism differs from both Banach and Kannan contractions [4]. This distinction is useful in generalized fuzzy spaces because different contractive mechanisms may require different combinations of t -norms, time parameters or auxiliary control functions.

3.5 Reich and Hardy-Rogers generalizations

Reich-type conditions combine direct and image-displacement terms. A standard schematic form is

$$d(Tx, Ty) \leq a d(x, y) + b d(x, Tx) + c d(y, Ty), \quad a+b+c < 1. \quad (5)$$

Hardy-Rogers-type conditions permit still more distance terms, including cross-distances [3,6]. Such inequalities are valuable for a review paper because they show that Banach, Kannan and Chatterjea contractions form part of a wider family. For the thesis, a unified E-fuzzy formulation may clarify which results are independent and which follow as corollaries of a more general theorem.

III. FUZZY SETS TO FUZZY METRIC SPACES

4.1 Fuzzy sets and t -norms

For a universe X , a fuzzy set A is characterized by a membership function $\mu_A: X \rightarrow [0, 1]$, where $\mu_A(x)$ measures the degree to which x belongs to A [9]. A continuous triangular norm (t -norm) is a binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ that is commutative, associative, monotone, continuous and satisfies $a * 1 = a$. Typical examples include the product t -norm $a * b = ab$ and the minimum t -norm $a * b = \min\{a, b\}$. T -norms are essential because they replace ordinary addition in the fuzzy triangle mechanism.

$$a * b = ab \quad \text{or} \quad a * b = \min\{a, b\}. \quad (6)$$

4.2 George-Veeramani fuzzy metric structure

In the George-Veeramani framework [11,12], a fuzzy metric $M: X \times X \times (0, \infty) \rightarrow (0, 1]$ represents the degree of nearness between x and y at a positive parameter t . The conditions include positivity, identity of indiscernibles, symmetry, a fuzzy triangle inequality and continuity in t . The key triangular relation has the form

$$M(x, z, t+s) \geq M(x, y, t) * M(y, z, s). \quad (7)$$

Here values closer to 1 represent greater nearness. Consequently, convergence is expressed by $M(x_n, x, t) \rightarrow 1$ for every $t > 0$. This inversion of the metric viewpoint is conceptually important: in ordinary metric spaces convergence means a distance tends to 0, whereas in fuzzy metric spaces it is commonly represented by nearness tending to 1.

$$x_n \rightarrow x \Leftrightarrow M(x_n, x, t) \rightarrow 1 \text{ for every } t > 0. \quad (8)$$

4.3 Fuzzy Cauchy sequences and completeness

A fuzzy Cauchy sequence is one whose terms become arbitrarily near in the fuzzy sense. A typical formulation requires that for each $\varepsilon \in (0, 1)$ and $t > 0$ there exists N such that $M(x_n, x_m, t) > 1 - \varepsilon$ whenever $n, m \geq N$. Completeness then means that every such Cauchy sequence converges to a point of the space. This structural layer is indispensable: generalized contraction theorems can only be proved rigorously after the exact convergence and completeness definitions have been fixed.

$$\forall \varepsilon \in (0, 1), \forall t > 0, \exists N: n, m \geq N \Rightarrow M(x_n, x_m, t) > 1 - \varepsilon. \quad (9)$$

4.4 Fixed point theory in fuzzy metric spaces

Grabiec [13] established influential fixed point results in fuzzy metric spaces, and Gregori and Sapena [14] further developed fixed point theorems in the George-Veeramani setting. Vasuki [15] extended the discussion to common fixed points for R -weakly commuting maps, while Mihet [16] examined fuzzy contractive mappings. These works demonstrate that the central architecture of classical fixed point proofs remains relevant, but the contraction

condition, completeness concept and continuity assumptions may require substantial reformulation.

V. MORINGA CONCEPTUAL EVOLUTION TOWARD E-FUZZY FIXED POINT THEORY

The development of the field may be read as a sequence of generalizations. Classical metric fixed point theory supplies contraction mechanisms; fuzzy set theory supplies graded membership; fuzzy metric spaces reinterpret closeness through membership-like nearness; and E-fuzzy structures broaden this setting further. The present research sits at the point where structural analysis, generalized contractions, common fixed points and robustness need to be integrated rather than treated as unrelated theorem families.

three-variable specialization. Since the supplied thesis seeks to build structural theory, it should state the exact specialization used for convergence and then prove basic limit lemmas before invoking them in fixed point proofs.

$E : X \times X \times X \times (0, \infty) \rightarrow [0, 1]$ (notation for a three-variable E-fuzzy nearness). (10)

6.2 A different E-fuzzy usage in the literature

Mecheraoui, Mitrovic, Parvaneh, Aydi and Saleem [26] use the expression E-fuzzy metric space for a distinct class of fuzzy metric spaces designed so that Banach and Meir-Keeler fixed point theorems can be obtained without some additional assumptions required in earlier fuzzy metric formulations. Their construction is based on a two-variable

Conceptual evolution and research architecture for E-fuzzy fixed point theory

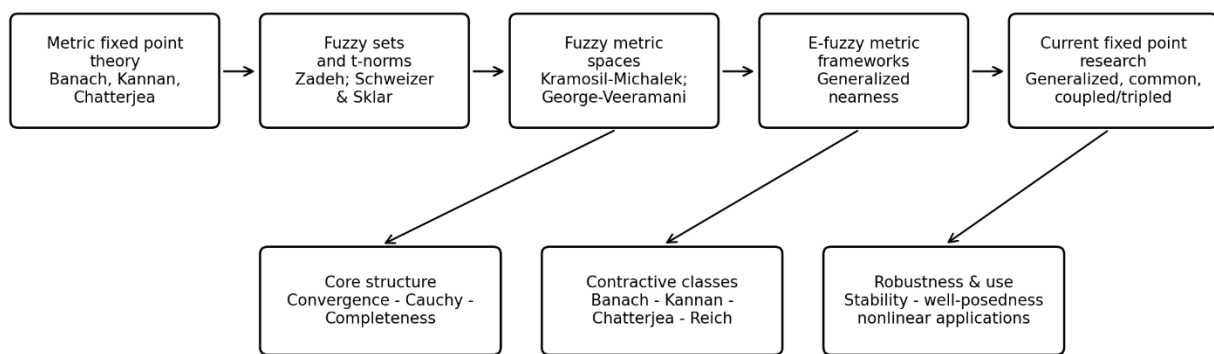


Figure 1. Conceptual evolution and research architecture of fixed point theory leading to E-fuzzy metric investigations.

Figure 1 also highlights the internal logic of the proposed thesis: structural definitions lead to contractive fixed point theorems; those theorems are extended to common fixed points; and finally stability, well-posedness and applications establish the robustness and practical value of the results.

VI. E-FUZZY METRIC SPACES: DEFINITIONS AND TERMINOLOGICAL DISCIPLINE

6.1 Generalized E-fuzzy metric line associated with Sukanya and Jose

Sukanya and Jose developed a generalized fuzzy metric framework and subsequently presented fixed point and contraction results in generalized E-fuzzy metric spaces [25,27,28]. In this line, the E-fuzzy metric is treated as a generalized fuzzy nearness structure involving three spatial variables and a positive parameter, conceptually related to G-metric-type generalization. The exact axioms used by an author must be reproduced before any new theorem is proved, because convergence and triangle-type arguments depend directly on those axioms.

For a three-variable E-fuzzy metric, the generic notation may be written as $E(x,y,z,t)$, with values in $[0,1]$ or $(0,1]$ depending on the adopted definition. A convergence relation is therefore expected to involve nearness of a sequence term and the limiting point in the appropriate

fuzzy metric and a uniformity/equicontinuity condition rather than the three-variable generalized structure used by Sukanya and Jose. This is not a minor notational difference; it changes the underlying object.

6.3 Implication for the present thesis

The thesis should therefore begin Chapter 3 with a “Definition Convention” stating exactly which E-fuzzy metric is adopted, followed by examples and non-examples. The phrase “E-component” should not be interpreted as an arbitrary transformation unless the chosen source definition actually introduces such an operator. Keeping the axiom system explicit will prevent the literature review, theorem statements and applications from mixing non-equivalent frameworks.

VII. GENERALIZED CONTRACTIVE MAPPINGS IN THE E-FUZZY SETTING

7.1 Guiding principle for fuzzy contraction design

In a fuzzy metric, greater nearness corresponds to values nearer 1. Consequently, a fuzzy contraction is often expressed by requiring the images to be “more near” than the original points after a suitable change of time scale or control function. There is no single universal formula that represents every fuzzy contraction. The correct E-fuzzy condition must be chosen so that the axioms permit an iterative estimate strong enough to prove E-Cauchy behavior.

A useful schematic control-function form is

$$E(Tx, Ty, Tz, t) \geq \Phi(E(x, y, z, \psi(t))), \quad (11)$$

where Φ is increasing and satisfies $\Phi(u) > u$ for $u \in (0, 1)$, while ψ controls the time parameter. Equation (11) is presented as a research template rather than as a theorem quoted from a single source. Any final theorem must specify Φ , ψ and the E-fuzzy axioms precisely and then verify that the iteration drives the nearness values to 1.

7.2 E-fuzzy Banach-type research form

A Banach-type E-fuzzy contraction should preserve the direct comparison between the images and the original points. One common strategy in fuzzy metrics is to change the time parameter by a factor $k \in (0, 1)$, so that the images are required to be at least as near at a shorter time scale as the original points are at t . A schematic three-variable version is

$$E(Tx, Ty, Tz, k t) \geq E(x, y, z, t), \quad 0 < k < 1. \quad (12)$$

The proof program is then to apply (12) repeatedly along Picard iterates, combine the resulting inequalities with the E-fuzzy triangle property, establish E-Cauchy behavior, and use completeness. This structure aligns directly with Hypotheses H1 and H2, but the theorem must show that its assumptions are genuinely weaker or broader than an existing fuzzy metric theorem before claiming improvement.

7.3 E-fuzzy Kannan-type research form

The Kannan mechanism depends on the relation of each point to its image. In a three-variable E-fuzzy setting this suggests that the contractive bound should depend on nearness terms containing x and Tx and terms containing y and Ty rather than only $E(x, y, \cdot, t)$. One possible research template is to combine self-displacement nearness through the t -norm:

$$E(Tx, Ty, Ty, t) \geq \Psi(E(x, Tx, Tx, t) * E(y, Ty, Ty, t)). \quad (13)$$

Here Ψ must be chosen so that the right-hand side produces strict improvement in nearness. Equation (13) is intentionally schematic. Its role is to preserve the logical content of Kannan's inequality (3), not to claim a ready-made theorem. A rigorous thesis contribution would define Ψ , determine an admissible parameter range and prove existence and uniqueness without silently assuming continuity if it is unnecessary.

7.4 E-fuzzy Chatterjea-type research form

A Chatterjea analogue must preserve cross interactions, replacing the metric terms $d(x, Ty)$ and $d(y, Tx)$ by E-fuzzy nearness terms that link each point to the image of the other. A schematic form is

$$E(Tx, Ty, Ty, t) \geq \Omega(E(x, Ty, Ty, t) * E(y, Tx, Tx, t)). \quad (14)$$

The main mathematical challenge is to select Ω and the time scaling so that the iterative sequence can be estimated without over-strong assumptions. A successful result would be especially valuable if it applies to a mapping that is neither an E-fuzzy Banach contraction nor an E-fuzzy Kannan contraction, supported by a carefully verified example.

7.5 Generalized and controlled contractions

The wider fixed point literature provides several mechanisms for strengthening generality: altering distance

functions [17], integral-type contractions [18], F-contractions [19] and simulation functions [20]. These methods suggest a pathway for E-fuzzy research in which the contraction is mediated by an auxiliary function rather than a constant. The advantage is that a broad theorem may contain Banach-, Kannan- and Chatterjea-type results as corollaries. The risk is excessive abstraction without demonstrable gain; therefore, every generalized theorem should be accompanied by a reduction map showing the classical or fuzzy result recovered under special choices of the control functions.

VIII. REVIEW OF FIXED POINT RESULTS SPECIFICALLY RELATED TO E-FUZZY STRUCTURES

Sukanya and Jose's 2018 work on generalized fuzzy metric spaces and generalized E-fuzzy fixed point theory provides an early basis for the three-variable strand [25, 27]. Their later contraction paper explicitly develops properties and a contraction theorem in E-fuzzy metric space [28]. These contributions establish that a fixed point theory can be built on the generalized E-fuzzy structure, but they also leave room for systematic comparison among classical contraction mechanisms and for a deeper structural treatment of convergence and completeness.

Mecheraoui et al. [26] address a different question: they identify a class of fuzzy metric spaces in which Banach and Meir-Keeler principles can hold without some additional conditions required by earlier fuzzy metric extensions. Their result is relevant to the present thesis conceptually because it demonstrates the value of weakening auxiliary assumptions. However, because their "E-fuzzy" definition is not the same as the generalized three-variable structure, direct theorem transfer requires caution.

Shahana and Jose [29] extend the three-variable E-fuzzy line to coupled coincidence/fixed point results under a ϕ -contraction and include an integral-equation application. Their 2025 work develops tripled fixed point theorems in complete E-fuzzy metric spaces and again connects the theory to integral equations [30]. These studies indicate that higher-dimensional fixed point concepts are active within E-fuzzy research and provide evidence that the framework can support nonlinear applications.

Kumar and Manjeet [31] introduce a generalized E-fuzzy b-metric space and establish common fixed point results for three pairs of self-maps. Their work illustrates another layer of generalization in which the triangle mechanism itself is relaxed. For the present thesis, this literature is useful as a boundary marker: the principal task is first to develop rigorous results in the chosen E-fuzzy metric space, while E-fuzzy b-metric variants can be treated as future extensions unless they are explicitly incorporated into the scope.

IX. RESEARCH GAPS AND FUTURE DIRECTIONS

The For two self-maps $S, T: X \rightarrow X$, a common fixed point x^* satisfies $Sx^* = Tx^* = x^*$. Common fixed point theory is essential when a nonlinear problem contains more than one operator. Classical work gradually weakened the interaction assumptions placed on the maps. Jungck [21] developed commuting and related mapping concepts, while Jungck and Rhoades [22] formalized weak compatibility in fixed point settings. Property E.A. and related notions were

also introduced to reduce dependence on strict commutativity [23].

$$Sx^* = Tx^* = x^*. \quad (15)$$

In an E-fuzzy metric space, the proof normally requires two components: an E-fuzzy contractive relation linking S and T, and an interaction condition that allows a coincidence point to become a common fixed point. Weak compatibility is attractive because it only requires commutativity at coincidence points rather than everywhere. Thus a major research question is whether continuity or stronger compatibility assumptions in existing E-fuzzy results can be replaced by weak compatibility without sacrificing existence or uniqueness.

The thesis objective concerning pairs of compatible or weakly compatible mappings is therefore well motivated. The strongest contribution would not merely restate a known fuzzy common fixed point theorem in E-fuzzy notation; it would identify exactly where the E-fuzzy structure permits weaker assumptions, prove the result, and construct an example in which the new theorem applies while the earlier theorem does not.

X. STABILITY AND WELL-POSEDNESS OF FIXED POINT PROBLEMS

10.1 Iterative stability

Fixed point existence is only one part of a useful approximation theory. In applications, computed iterates are perturbed by rounding, measurement error or approximate operator evaluation. Stability asks whether such perturbations preserve convergence to the intended fixed point. Harder and Hicks [32] and Berinde [33] provide important classical treatments of iterative stability. An E-fuzzy stability concept should measure whether perturbed iterates remain increasingly near to the exact iterative orbit and ultimately to the same fixed point.

If $x_{n+1}=T(x_n)$ is the exact iteration and y_n is an approximate sequence, a metric residual is commonly represented by $d(y_{n+1}, T y_n)$. In an E-fuzzy setting, the residual must be reformulated as high nearness between y_{n+1} and $T y_n$. A possible qualitative condition is

$$E(y_{n+1}, T y_n, T y_n, t) \rightarrow 1 \text{ as } n \rightarrow \infty. \quad (16)$$

A stability theorem would then show that the residual condition in (16), together with the E-fuzzy contraction, implies $E(y_n, x^*, x^*, t) \rightarrow 1$. Such a result would directly address a gap identified in the thesis draft and would make the fixed point theory more relevant to numerical approximation under uncertainty.

10.2 Well-posedness

Well-posedness of a fixed point problem generally requires existence and uniqueness of the fixed point and convergence of every approximate fixed point sequence to that point. Reich and Zaslavski [34] developed influential metric-space formulations. In an E-fuzzy framework, an approximate fixed point sequence $\{x_n\}$ may be characterized by E-fuzzy nearness between x_n and $T x_n$ tending to 1:

$$E(x_n, T x_n, T x_n, t) \rightarrow 1 \Rightarrow E(x_n, x^*, x^*, t) \rightarrow 1. \quad (17)$$

The implication in (17) is a natural target for theorem development, but its validity depends on the exact E-fuzzy axioms and contraction condition. A systematic well-posedness chapter should therefore state the definition, prove it for each major contractive class or for a unified generalized contraction, and give an example illustrating the approximation mechanism.

XI. APPLICATIONS TO NONLINEAR MATHEMATICAL PROBLEMS

The practical significance of a fixed point theorem is best demonstrated by reducing a nonlinear problem to an operator equation $x=T(x)$. Integral equations provide a natural test case because recent E-fuzzy literature already uses them [29,30]. Consider a nonlinear Fredholm-type equation

$$x(t) = g(t) + \int_a^b K(t,s,x(s)) ds. \quad (18)$$

Define an operator T on a suitable function space by

$$(Tx)(t) = g(t) + \int_a^b K(t,s,x(s)) ds. \quad (19)$$

A solution of the integral equation is exactly a fixed point of T. The E-fuzzy application problem is therefore to choose a fuzzy metric on the function space, verify completeness, and show that the kernel condition makes T satisfy one of the new E-fuzzy contraction hypotheses. This converts an abstract theorem into an existence-and-uniqueness result for the nonlinear equation.

The same strategy applies to differential equations after an equivalent integral formulation, nonlinear operator equations, iterative optimization schemes and equilibrium problems. However, an application should not be added merely as an illustration: all space assumptions, mapping properties and constants must be checked explicitly. For a doctoral thesis, at least one application should use a genuinely nontrivial nonlinear operator for which the generalized E-fuzzy theorem offers a demonstrable advantage over a classical contraction argument.

XII. COMPARATIVE SYNTHESIS OF MAJOR CONTRACTIVE AND E-FUZZY DIRECTIONS

Table 1 summarizes the logical mechanism of the most relevant contraction classes and their relationship to the proposed E-fuzzy study. The comparison shows that the thesis can be organized around preservation of mechanism: direct contraction for Banach, self-displacement for Kannan, cross-displacement for Chatterjea, weighted combinations for Reich-type mappings, and interaction conditions for common fixed points.

Table 1. Comparative map of fixed point mechanisms and their relevance to E-fuzzy research.

Refere nce / class	Core mechani sm	Typical assumptio ns	Main conclusion	Relevance to thesis
Banac h [1]	Direct distance reductio n	Completen ess + $\lambda < 1$	Unique fixed point; Picard convergence	Direct E- fuzzy nearness improvm ent

Reference / class	Core mechanism	Typical assumptions	Main conclusion	Relevance to thesis
Kannan [2]	Self-displacement terms	$\alpha < 1/2$; completeness	Unique fixed point	E-fuzzy contraction based on $x \leftrightarrow Tx$ and $y \leftrightarrow Ty$
Chatterjea [4]	Cross-displacement terms	$\beta < 1/2$; completeness	Unique fixed point	E-fuzzy cross-nearness formulation
Reich [3] / Hardy-Rogers [6]	Weighted multiple terms	Coefficient restrictions	Generalized uniqueness/existence	Potential unifying E-fuzzy theorem
Fuzzy metric [13,14,16]	Nearness tends to 1	Fuzzy completeness + contractive condition	Fuzzy fixed points	Immediate theoretical background
Common fixed points [15,21-23]	Interaction of two or more maps	Compatibility/weak compatibility etc.	Coincidence /common fixed points	Objective H3
Sukanya-Jose E-fuzzy [25,27,28]	Generalized three-variable fuzzy nearness	Chosen E-fuzzy completeness and contraction	Fixed point/contraction results	Primary structural line for thesis
Meche-raoui et al. [26]	Special class of two-variable fuzzy metrics	Uniformity property	Banach/Meir-Keeler without some extra assumptions	Terminology caution; assumption-reduction insight
Shahana-Jose [29,30]	Coupled/tripled E-fuzzy fixed points	ϕ -contraction; complete E-fuzzy space	Higher-order fixed points + integral equations	Evidence of active applications
Kumar-Manjeet [31]	Generalized E-fuzzy b-metric	Relaxed metric structure; multi-map conditions	Common fixed points for three pairs	Future/generalized extension

XIII. CRITICAL ANALYSIS AND RESEARCH GAPS

The literature demonstrates substantial development but also several identifiable gaps that justify a focused thesis on fixed point theory in E-fuzzy metric spaces.

- Definitional consistency remains essential. The label E-fuzzy metric space is used for non-equivalent structures,

so a theorem cannot be compared across papers unless the axioms are first aligned.

- The structural layer requires systematic treatment. Convergence, E-Cauchy sequences, completeness, uniqueness of limits and continuity properties should be proved before advanced fixed point theorems are developed.
- Banach-type results are more visible than a unified comparative theory of Banach-, Kannan- and Chatterjea-type contractions in the same E-fuzzy framework. The relative strength and independence of these conditions therefore deserve careful analysis.
- Common fixed point results remain an opportunity for assumption reduction. In particular, replacing continuity, commutativity or strong compatibility by weaker interaction conditions may generate meaningful improvements.
- Stability and well-posedness are much less developed in the specific E-fuzzy literature than basic existence and uniqueness. This gap aligns directly with the sixth objective of the thesis.
- Examples need to do more than verify axioms. Strong examples should separate theorem classes by showing a mapping that satisfies the new hypothesis but fails a known stronger hypothesis.
- Applications should verify the E-fuzzy contraction for a concrete nonlinear operator, preferably an integral or operator equation, rather than merely state that applications are possible.

XIV. CONCLUSION

Fixed point theory has progressed through a sequence of increasingly flexible settings: classical complete metric spaces, generalized contraction classes, fuzzy metrics and more recent E-fuzzy structures. The literature reviewed here confirms that E-fuzzy fixed point theory is sufficiently developed to provide a foundation but sufficiently open to support substantial new work. In particular, systematic structural analysis, comparative Banach/Kannan/Chatterjea theory, common fixed points under weaker interaction conditions, stability, well-posedness and carefully verified applications remain productive directions.

For the thesis “A Study on Fixed Point Theory in E-Fuzzy Metric Spaces,” the most important methodological requirement is definitional precision. Once a single E-fuzzy framework is fixed, the research can proceed cumulatively from convergence and completeness to fixed point existence and uniqueness, generalized contractions, common fixed points, robustness and applications. The proposed objectives and hypotheses are well aligned with this program, provided that claims of generalization are demonstrated through explicit corollaries and separating examples rather than asserted only at a conceptual level.

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